M2 Project

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Abstract

In this text I give a summary of Horel's integral homotopy theory, and a brief survey of Nori motives. A relationship between the two is then established by formally stating two conjectures by Horel. Towards this goal, I construct a model structure for cosimplicial finitely generated modules over a binomial flat Hopf algebra compatible with those of Horel. This particularises to cosimplicial representations of the motivic Galois group ie. Nori motives.

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Contents

NOTATION AND CONVENTIONS

- The natural numbers $\mathbb{N}$ are the non-negative integers.
- By the word "space" I will mean a simplicial set.
- For a ring R, the category $R-\mathbf{mod}$ is the category of finitely generated $R-modules$, the category **Comod** is that of finitely generated R-comodules.
- Enriched Hom-sets will be underlined: $\underline{\mathrm{Hom}}(-,-)$. In the case that $\mathcal{C}$ is a simplicial category, I will denote by $\mathrm{map}_{\mathcal{C}}(-,-)$ its simplicially-enriched Hom-object.
- I will mean by R-localisation those whose weak equivalences are quasi-isomorphisms in R-coefficients.
- Given a model category $\mathcal{M}$, we will represent cofibrations by the arrows $\hookrightarrow$, fibrations by arrows $\twoheadrightarrow$, weak equivalences by $\xrightarrow{\sim}$, and weak (co)fibrations by a combination of these.
- Given an object A in a model category $\mathcal{M}$, the superscripts A^c and A^f will denote a cofibrant and fibrant replacement respectively.
- Given a category $\mathcal{C}$, I will write $\mathbf{c}\mathcal{C} := \mathbf{Fun}(\Delta, \mathcal{C})$ for the category of cosimplicial objects in $\mathcal{C}$.
- Given a functor F, I will denote its left derived functor $\mathbb{L}F$ and its right derived $\mathbb{R}F$.
- I will use homological grading, in the sense that given the n^{th} entry C_n of a chain complex C_*, the n^{th} entry of its dual is $(C^{\mathrm{op}})^{-n}$. In particular, the chain $\mathbb{Z}[n]$ with $\mathbb{Z}$ in its n^{th} entry and 0 elsewhere will have dual $\mathbb{Z}[-n]$.

1 A SORT OF INTRODUCTION

1.1 STORYTELLING

This story revolves around *Binomial Rings and Homotopy Theory* by Horel [Hor22] and Nori motives.

In the rational world, Sullivan constructed an adjuction between the homotopy category of simplicial sets and that of differential-graded commutative $\mathbb{Q}$-algebras

$$\Omega^*_{PL} : \mathbf{Ho}(\mathbf{sSet}) \rightleftharpoons \mathbf{Ho}(\mathbf{dgcAlg}^{\mathrm{op}}_{\mathbb{Q}}) : \langle - \rangle$$

so that when restricted to connected nilpotent finite type rational spaces, the functor Ω^*_{PL} becomes fully faithful.

The problem comes, as usual, when we replace the rationals by the integers. Both Mandell [Man06] and Töen [**Toen20**] found equivalent versions the adjuntion of the form $\mathbf{Ho}(\mathbf{sSet}) \rightleftharpoons \mathbf{Ho}(\mathbf{Ring})$ whose left adjoint $X \mapsto \mathbb{Z}^X$ is faithful when restricted connected nilpotent spaces of finite type, but fails to be full. Horel's idea is to give more structure to the category of rings to make it also full. This more structured class of rings are *binomial rings*.

Horel, then, produces an adjunction

$$\mathbb{Z}^{(-)} : \mathbf{Ho}(\mathbf{sSet}) \rightleftharpoons \mathbf{Ho}(\mathbf{cBRing}^{\mathrm{op}}) : \langle - \rangle$$

such that:

Theorem A. Let X be a connected nilpotent finite type space, then we have the following weak equivalence $X \xrightarrow{\sim} \langle \mathbb{Z}^{(X)} \rangle$. In particular, the functor $\mathbb{Z}^{(-)}$ is fully faithful when restricted to connected nilpotent finite type spaces.

Theorem B. Let X be a space, then the unit of the adjunction $\langle \mathbb{Z}^X \rangle$ is weakly equivalent to its Bousfield-Kan $\mathbb{Z}$-completion $\mathbb{Z}_\infty X$.

This adjunction mimics the behaviour of Sullivan's functor Ω_{PL} in the rational setting.

Our parallel story is that of the theory of motives. It famously dates back to Grothendieck, whose goal was to prove the Weil conjectures using this category of universal (co)homology theories. Although the existence of such category is still conjectural, we have some candidates. One of these was constructed by Nori [Nor] in a purely algebro-categorical manner that relies on a weak version of Tannaka duality. This category is equivalent to the representations of an affine algebraic group scheme, the *motivic Galois group*, which is just the category of comodules over a Hopf algebra. Interestingly enough, Nori's Hopf algebra H^N is binomial as a ring, which allows us to bridge the gap between integral homotopy theory and Nori motives.

Vague Conjecture. Integral homotopy groups of varieties over a field $k \hookrightarrow \mathbb{C}$ are Nori motives.

The sceptical reader might rest assured that this conjecture shall be stated rigorously in the text.

This brief paper adds some modest yet honest work towards this result by giving a model structure compatible with those of integral homotopy theory to categories to categories of cosimplicial objects in Nori motives.

1.2 ORGANISATION OF THE PAPER

In section 2 of the paper I will review Horel's construction of integral homotopy theory. I will put some emphasis in explaining the case of non-nilpotent spaces and their R-completion, especially on the relationship with the original work of Bousfield and Kan [BK72], and the modern approach by Isaksen [Isa05] that Horel uses.

In section 3, I will give a brief overview of Nori motives. First, I present Nori's original approach, explaining the motivation behind his universal construction; an otherwise seemingly *ad hoc* construction. In the second part, I give a more modern definition of Nori motives based on conjectural grounds; mostly for technical reasons.

Section 4 of the paper will commence by finally making precise Horel's conjectures on the relationship between Nori motives and integral homotopy theory. The second part is dedicated to putting cofibrantly generated model structures compatible with those of integral homotopy theory. To do this, I will use results of Hovey [Hov07] on models on abelian categories, which I review. Finally, I present the main "original" result of the text:

Proposition 4.5. There exists an abelian cofibrantly generated model structure on the category of cosimplicial comodules which are levelwise binomial rings over a flat binomial Hopf algebra H, $\mathbf{cBRing}_H$, where **fibrations** are **epimorphisms**, and **weak equivalences** are **quasi-isomorphisms**.

In particular, we get that the category of cosimplicial binomial Nori motives admits the above model structure.

2 INTEGRAL HOMOTOPY THEORY À LA HOREL

2.1 EXTRA STRUCTURE: BINOMIALITY

Definition 2.1. A commutative ring A is said to be **binomial** if its underlying group is torsion free, and for all primes p in A in $\mathbb{Z}$ and for all a in A we have

$$a^p \equiv a \text{ Mod } pA$$

- 2.1.1. Alternative equivalent definitions of binomial rings can be found in [Xan17]. We denote the full subcategory of binomial rings by **BRing**.

There exists a forgetful functor U : **BRing** → **Ring** with both a left and right adjoint; we will be more interested in its left adjoint:

$$\mathrm{Bin}^U : \mathbf{Ring} \to \mathbf{BRing}$$

where for a ring A, $\mathrm{Bin}^U(A)$ is the intersection of all binomial subrings of $A \otimes_{\mathbb{Z}} \mathbb{Q}$ containing the image of A in $A \otimes_{\mathbb{Z}} \mathbb{Q}$.

- 2.1.2. As U as both a left and a right adjoint, preserves limits and colimits, in particular, it satisfies the conditions for Beck's monadicty theorem. Hence, we can see **BRing** as the category of algebras over the monad $U \circ \mathrm{Bin}^U$.

- 2.1.3. In the concrete case that S is a set, and $\mathbb{Z}[S]$ the ring of polynomials with variables in S, we can define $\mathrm{Num}[S] := \mathrm{Bin}^U(\mathbb{Z}[S])$, the **numerical polynomials in S variables**. When the variables are just $x_1, x_2, ..., x_n$ we write $\mathrm{Num}[x_1, ..., x_n]$. Observe that $\mathrm{Num}[x]$ has a cocommutative coalgebra structure given by

$$\begin{gathered}\Delta : \mathrm{Num}[x] \to \mathrm{Num}[x] \otimes \mathrm{Num}[x] \cong \mathrm{Num}[x, y] \\ f \mapsto \Delta f(x, y) = f(x + y)\end{gathered}$$

- 2.1.4. A relevant property of the binomial ring $\mathrm{Num}[x]$ that will be appear in the proof of our main theorem A is that its cobar construction is quasi-isomorphic to S^1. In the case of C^* the constant coalgebra $\mathrm{Num}[x]$, its cobar construction will simply be $\Omega(\mathrm{Num}[x])^n = \mathrm{Num}[x]^{\otimes n}$.

Proposition 2.1. *[Hor22, Proposition 2.1] The cohomology of $\Omega(\mathrm{Num}[x])$ is free of rank 1 in cohomological degree 0 and 1 and is zero in any other degree.*

Proof. The key is dualising: as the cobar construction is levelwise free and of finitely generated ($\mathrm{Num}[x]^{\otimes n}$), we can dualise by $\mathrm{Hom}(-, \mathbb{Z})$ and get a graded abelian group whose homology is the same as the cohomology of the cobar construction. The homology of this dualised bar construction is well known and coincides with our desired result. □

Going back to our adjunction **BRing** ⇌ **Ring**, we can upgrade this adjunction to the cosimplicial setting. By abuse of notation we get

$$\mathbb{Z}^{(-)}\mathbf{cRing} \rightleftharpoons \mathbf{cBRing} : \mathrm{Bin}^U$$

$$(X)^n \mapsto ([n] \mapsto \mathrm{Bin}^U(X^n))$$

We want to give these categories, especially that of cosimplicial binomial rings, a model structure. We take advantage of the model structure given to **cAb** by the Dold-Kan correspodence; to do so:

Construction 2.2. The category **cAb** has a transferred model structure from the category of cochain complexes $\mathbf{Ch}^\bullet(\mathbb{Z})$ where:

- **fibrations** are levelwise **epimorphisms** of abelian groups.
- **weak equivalences** are maps that are sent to **quasi-isomorphisms** in $\mathbf{Ch}^\bullet(\mathbb{Z})$.

Moreover, this model structure is simplicial with cotensoring given by

$$\mathbf{cAb} \times \mathbf{sSet}^{\mathrm{op}} \to \mathbf{cAb}$$

$$(A, X) \mapsto B^X := [p] \mapsto (A^p)^{X_p}$$

where A is a cosimplicial abelian group and X is a simplicial set.

Theorem 2.2. *There is a simplicial model structure in* **cRing** *where* ***fibrations*** *are* ***epimorphisms*** *of the underlying abelian groups, and* ***weak equivalences*** *are* ***quasi-isomorphisms****.*

Proof. We use a transfer theorem along the right adjoint of $\mathbf{cAb} \rightleftharpoons \mathbf{cRing}$.

Let A be a cosimplicial abelian group, we define $A^I := A^{\Delta^1}$, the cotensoring with the simplex represented by $[1]$. For this to be a path object it must satisfy the following

$$A \xrightarrow{\sim} A^I \twoheadrightarrow A \times A$$

It is clear that the second arrow is a levelwise surjection, so an epimorphism. As the weak equivalences are defined to be those in $\mathbf{Ch}^*(cAb)$, it is sufficient to see that $N(X^I) \xrightarrow{\sim} \mathrm{NX}^{NI}$, where NI is the interval element in $\mathbf{Ch}(\mathbf{cAb})$: $\mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \to 0 \to \ldots \to 0 \to \ldots$ This is the path object in connective cochains, so we get the desired weak equivalence.

We can now apply the *path object argument*: a standard procedure to transfer a model structure along a right adjoint if all objects in the category are fibrant and they have a functorial path object (for our case see [SS00, Theorem A.3][1]). □

The interesting property about the above constructed path object A^I is that it will also be a path object in **cBRing**, which allows to also give it a transferred model structure, as the model structure only depends on morphisms between cochain complexes of abelian groups.

Theorem 2.3. *There is a model structure on* **cBRing** *in which*

- *The* ***weak equivalences*** *are the* ***quasi-isomorphisms****.*
- *The* ***fibrations*** *are the maps that are* ***degreewise epimorphisms*** *of abelian groups.*

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[1] Beware, there exists a version of this paper where theorem A.3 is labelled lemma 2.3.

Proof. We use the path object argument. □

- 2.1.5. Moreover, the adjunction $\mathrm{Bin}^U : \mathbf{cRing} \rightleftharpoons \mathbf{cBRing} : U$ is a Quillen adjunction.

- 2.1.6. The category **cBRing** inherits a simplicial model structure from **cAb**. Particularly, it has a cotensoring B^X for a cosimplicial binomial ring B and a simplicial set X. This allows us to define the following functor by fixing $B = \mathbb{Z}$

$$\begin{aligned} \mathbb{Z}^{(-)} : \mathbf{sSet} &\to \mathbf{cBRing}^{\mathrm{op}} \\ X &\mapsto \mathbb{Z}^X \end{aligned}$$

Where $\mathbb{Z}^X$ is degreewise a product of copies of $\mathbb{Z}$. This functor has a derived right adjoint defined as

$$A \mapsto \langle A \rangle := \mathrm{map}_{\mathbf{cBRing}}(A^c, \mathbb{Z})$$

where A^c is a cofibrant replacement of A.

2.2 The Nilpotent Case

- 2.2.1. We now turn our attention to "tame" simplicial sets: Eilenberg-MacLane spaces. Let $\mathbb{Z}[n]$ be the chain with $\mathbb{Z}$ in its n^{th} level and 0 everywhere else, we can define the $(\mathbb{Z}, n)$ **Eilenberg-MacLane space** K_n to be the simplicial set associated to $\mathbb{Z}[n]$ by the Dold-Kan correspondence. We now establish an equivalence between the two functors $\mathbb{Z}^{(-)}$ and $\mathrm{Bin}^U \circ \mathrm{Sym} \circ \mathrm{Hom}(-, \mathbb{Z}) \circ \mathbb{Z}\langle - \rangle : \mathbf{sSet} \to \mathbf{cBRing}^{\mathrm{op}}$ for Eilenberg-MacLane spaces.

The following concatenation of adjunctions will prove itself useful:

$$\begin{tikzcd}
\mathbf{sSet} \arrow[r, shift left, "\mathbb{Z}\langle\rangle"] & \mathbf{sAb} \arrow[l, shift left] \arrow[r, shift left, "\mathrm{Hom}(-{,}\mathbb{Z})"] \arrow[d, shift left, "N_\bullet"] \arrow[d, phantom, "\simeq"] & \mathbf{cAb}^{\mathrm{op}} \arrow[l, shift left] \arrow[r, shift left, "\mathrm{Sym}(-)"] & \mathbf{cRing}^{\mathrm{op}} \arrow[l, shift left] \arrow[r, shift left, "\mathrm{Bin}^U"] & \mathbf{cBRing}^{\mathrm{op}} \arrow[l, shift left] \\
& \mathbf{Ch}_\bullet(\mathbb{Z}) \arrow[u, shift left, "\Gamma"] & & &
\end{tikzcd}$$

where:

- $N : (C)_n \mapsto \left(C_n / \bigoplus_{i \le n} \mathrm{Im}(s_i)\right)_n$
- $\mathbb{Z}\langle - \rangle : (X)_n \mapsto \mathbb{Z}\langle X_n \rangle$
- $\mathrm{Sym}(-) : (A)^n \mapsto \bigoplus_{i \in \mathbb{N}} ((A^n)^{\otimes i} / \Sigma_i)$

We get to our desired relationship by:

1. By the Dold-Kan correspondence, we obtain the Eilenberg-MacLane space K_n associated to the chain complex $\mathbb{Z}[n]$, with $\mathbb{Z}$ in the n^{th} degree and 0 elsewhere.

2. By the counit map of the free-forgetful adjunction $\mathbf{sSet} \rightleftarrows \mathbf{sAb}$ we obtain $\mathbb{Z}\langle K_n \rangle \to \mathbb{Z}[n]$ in $\mathbf{sAb}$.
3. We apply the duality functor $\mathrm{Hom}(-, \mathbb{Z})$ and get the map $\mathbb{Z}[-n] \to \mathbb{Z}^{K_n}$ in $\mathbf{cAb}$.
4. We apply the adjuctions $\mathbf{cAb} \rightleftarrows \mathbf{cRing} \rightleftarrows \mathbf{cBRing}$ to get the map

$$\alpha_n : \mathrm{Bin}^U(\mathrm{Sym}(\mathbb{Z}[-n])) \to \mathbb{Z}^{K_n}$$

- 2.2.2. From now on, we shall denote $\mathrm{Bin}^U(\mathrm{Sym}(\mathbb{Z}[-n]))$ by F_n.

Proposition 2.4. *For all $n \in \mathbb{N}$, the map $\alpha_n : F_n \to \mathbb{Z}^{K_n}$ is a weak equivalence.*

Proof. We prove this by induction:

Base case: $n = 1$.

By the inverse Dold-Kan correspondence, we start with the simplicial abelian group $\mathbb{Z}[1]$ associated to the chain complex that is $\mathbb{Z}$ on degree 1 and 0 elsewhere. Recall that for a chain complex C_*, this is defined as $\Gamma(C_*)_n := \bigoplus_{[n] \twoheadrightarrow [k]} C_k$; so in our case we get

$$[n] \mapsto \mathbb{Z}^n$$

With face maps given by the addition of two consecutive interior faces, and 0 for the exterior ones. We now dualise to get the cosimplicial abelian group $\mathbb{Z}[-1]$. It is levelwise the same, and the coface maps are given by the diagonal map for interior faces and the 0 map for exterior ones. Hence, when we apply the functor $\mathrm{Bin}^U \mathrm{Sym}(\mathbb{Z}[-])$, we get the cosimplicial ring

$$[n] \mapsto \mathrm{Num}[x]^{\otimes n}$$

This is the cobar construction of $\mathrm{Num}[x]$. We know by 2.1 that the cohomology of this object coincides with that of the 1-sphere, which is that of the Eilenberg-MacLane space K_1. One can then show that this isomorphism is induced by α_1.

Inductive step:

Note that the functor $\mathrm{Bin}^U \circ \mathrm{Sym}$ is left Quillen, hence, its left derived functor commutes with homotopy colimits. Whence we get the following homotopy squares:

$$\begin{tikzcd}
\mathbb{Z}[-(n+1)] \arrow[r] \arrow[d] & 0 \arrow[d] & & F_{n+1} \arrow[r] \arrow[d] & \mathbb{Z} \arrow[d] \\
0 \arrow[r] & \mathbb{Z}[-n] & & \mathbb{Z} \arrow[r] & F_n
\end{tikzcd}$$

The first pushout diagram in $\mathbf{cAb}$ induces the second one via the functor $\mathrm{Bin}^U \circ \mathrm{Sym}(-)$. Let us denote the latter square by S_1.

We also have

$$\begin{tikzcd}
K_n \arrow[r] \arrow[d] & * \arrow[d] & & \mathbb{Z}^{K_{n+1}} \arrow[r] \arrow[d] & \mathbb{Z} \arrow[d] \\
* \arrow[r] & K_{n+1} & & \mathbb{Z} \arrow[r] & \mathbb{Z}^{K_n}
\end{tikzcd}$$

The first homotopy pullback square in **cAb** satisfies the conditions of theorem A.1 in [Toë20] (a generalisation of a result by Eilenberg-Moore) so we get that

$$\operatorname{map}_{\mathbf{cAb}}(*, \mathbb{Z}) \otimes^{\mathbb{L}}_{\operatorname{map}_{\mathbf{cAb}(K_{n+1}, \mathbb{Z})}} \operatorname{map}_{\mathbf{cAb}}(*, \mathbb{Z}) \xrightarrow{\sim} \operatorname{map}_{\mathbf{cAb}(K_n, \mathbb{Z})}$$

The term on the left is by definition the homotopy pushout of the diagram on the right, so we get the desired equivalence. Let us denote this second square by S_2.

We get induced maps from S_1 to S_2 via the α_n and α_{n+1}. We also know that α_n is equivalent to the pushout of the map α_{n+1}, which is given by the left derived tensor product $\mathbb{Z} \otimes^L_A \mathbb{Z}$, known as the Bar construction $B(A)$. Indeed, what we can prove that given that $\alpha_n = B(\alpha_{n+1})$ is a weak equivalence, so is α_{n+1}. Notice that normally we have results of the converse form: "given f_n a weak equivalence and $f_{n+1} = F(f_n)$ for some functor F, show that f_{n+1} is one too", however, in the appendix, Horel checks that the Bar constrcution is conservative, ie. it reflects isomorphims [Hor22, Theorem A.3], so we get the desired result that α_{n+1} is a weak equivalence. ☐

Proposition 2.5. *The unit map* $\eta_n : K_n \mapsto \langle \mathbb{Z}^{K_n} \rangle$ *is a weak equivalence*

Proof. We prove this by taking advantage of the previous proposition and the $2-3$ property of weak equivalences of model categories. Consider the composite map $\beta_n : K_n \to \langle \mathbb{Z}^{K_n} \rangle \to \langle F_n \rangle$

Where the first map of the composition is η_n and the second map is induced by α_n. Then, it suffices to show that the composite β_n is a weak equivalence for η_n to be one.

We can observe that

$$\langle F_n \rangle := \operatorname{map}_{\mathbf{cBRing}}(F_n, \mathbb{Z}) = \operatorname{map}_{\mathbf{cBRing}}(\operatorname{Bin}^U \operatorname{Sym}(\mathbb{Z}[-n]), \mathbb{Z}) = \operatorname{map}_{\mathbf{cAb}}(\mathbb{Z}[-n], \mathbb{Z})$$

Where the last identity is given by the free-forgetful adjunction. Thus, we deduce that $\langle F_n \rangle$ is a Eilenberg-MacLane space. It only remains to show that the map β_n is a weak equivalence between Eilenberg-MacLane spaces; this will follow from the following two claims:

1. A map $f : K_n \to X$ where X is an Eilenberg-MacLane space of type $(\mathbb{Z}, n)$, is a weak equivalence if and only if ι_n is in the image of $H^n(f)$. Where ι_n

denotes the class in $H^n(K_n, \mathbb{Z})$ corresponding to $1 \in \mathbb{Z}$ through the universal coefficient theorem and the Hurewicz isomorphism :

$$H^n(K_n, \mathbb{Z}) \simeq \mathrm{Hom}(\pi_n(K_n), \mathbb{Z}) \simeq \mathbb{Z}$$

2. The class ι_n is in the image of $H^n(\beta_n)$.

For claim 1, we observe that one of the directions is clear. Namely, if $H^n(f)$ is an isomorphism, it must contain the class of 1. Conversely, suppose ι_n were in $H^n(f)$, then $h(f)$ is surjective, and a surjective ring morphism $\mathbb{Z} \to \mathbb{Z}$ must be an isomorphism. We pass to $\mathrm{Hom}(\pi_n(K_n), \mathbb{Z})$ by Hurewicz to deduce that $\pi_n f$ is an isomorphism, and as all the other homotopy groups are 0, the map f must the an isomorphism.

Finally, to prove 2, note that the map β_n is adjoint to α_n by construction, and the class ι_n is in $H^n(\alpha_n)$. , We can construct another map $F_n \to \mathbb{Z}^{K_n}$ by letting

$$\gamma : F_n \to \mathbb{Z}^{\langle F_n \rangle} \to \mathbb{Z}^{K_n}$$

The first map is the adjunction $\mathbf{sSet} \rightleftharpoons \mathbf{cBRing}^{\mathrm{op}}$, and the second one is induced by β_n. The map γ_n is chain homotopic to α_n, so $H^n(\gamma_n)$ also contains the class ι_n, which suffices for the proof. □

- 2.2.3. Horel then gives a series of five properties that when satisfied by a set U of spaces imply that such set contains all connected nilpotent spaces of finite type [Hor22, Lemma 5.3]. Heuristically, these five conditions amount to all the EIlenberg-MacLane spaces of finitely generated groups being in the set, and closure regarding weak equivalence, Cartesian products, homotopy limits of towers, and fibers of maps with simply connected base.

Theorem A. [Hor22, Theorem 5.4] The derived unit map $X \mapsto \langle \mathbb{Z}^X \rangle$ is a weak equivalence for all connected nilpotent finite type spaces X.

Proof. The following strategy suffices to prove the proposition: let U be the set of all spaces X satisfying the property "$X \mapsto \langle \mathbb{Z}^X \rangle$ is a weak equivalence", if we prove that U satisfies the conditions mentioned in paragraph 2.2.3, then all connected nilpotent finite type spaces will be in U; that is, the derived unit map restricted to them is a weak equivalence.

□

2.3 GENERAL CASE: BOUSFIELD-KAN $\mathbb{Z}$-COMPLETION

Although we have already achieved the goal of producing a fully faithful functor on nilpotent spaces, our unit $X \mapsto \langle \mathbb{Z}^X \rangle$ will also behave in a nice way on general spaces: it will be its Bousefield-Kan $\mathbb{Z}$-completion.

In this section we will review the Bousfield-Kan R-completion of a simplicial set X, given its relevance in Horel's paper. Although this construction first appeared in [BK72], a more modern treatment will be followed, making use of pro-objects and categories following [Isa05].

- 2.3.1. Indeed, the Bousfield-Kan R-completion is a “homotopical version” of Malcev nilpotent completion, in the sense that the homotopy groups of $R_\infty X$ will be the nilpotent completion of the homotopy groups of X a nilpotent space.Bousfield-Kan R-completion gives an R-localisation that interacts "nicely" with homotopy: for simply connected spaces X, Y:[BK72, Proposition 3.2]

- $f : X \to Y$ induces a homotopy equivalence of $R_\infty X \simeq R_\infty Y$

  if and only if
- it induces and equivalence of their R-homotopy groups $R \otimes \pi_* X \cong R \otimes \pi_* Y$ if and only if
- it induces an R-homology isomorphism $H_*(X, R) \cong H_*(Y, R)$.

Recall, though, that we are precisely interested in the non-nilpotent space, so the above nice properties. However, it is nice to know that our derived unit map, which is a weak equivalnece $X \mapsto \langle \mathbb{Z}^X \rangle$ satisfies the same conditions as the $\mathbb{Z}-$completion.

For coming use, we recall the definition of a Postnikov tower:

Definition 2.3. Let X be a space, a **Postnikov tower** for X is a sequence of spaces $...P_n X \xrightarrow{q_n} P_{n-1}X \to ... \xrightarrow{q_0} P_0 X$, called the **Postnikov sections**, and maps $p_n : X \to P_n X$ for all $n \in \mathbb{N}$ such that:

- $P_n X$ is $n-$truncated and the map $p_n : X \to P_n X$ induces an isomorphism $\pi_i(X, x) \cong \pi_i(P_n X, x)$ for all $i \leq n$ and $x \in P_0 X$.
- The following diagram commutes for all $n \in \mathbb{N}$

$$\begin{tikzcd}
& X \arrow[dl, "i_{n-1}"'] \arrow[dr, "i_n"] & \\
P_{n-1}X \arrow[rr, "q_n"'] & & P_n X
\end{tikzcd}$$

Construction 2.4. Given a space X, the original construction of its R-completion goes briefly like this [BK72, Ch.1]:

1. We are given a monad $R : \mathbf{sSet} \to \mathbf{sSet}$, and we define a space $(R \otimes X)_n := \Pi_{X_n} R$.
2. We define the subsimplicial set RX as those simpleces $\Sigma r_i x_i$ in $R \otimes X$ such that $\Sigma r_i = 1$, and we choose a base point for the space. More generally, can define $R^k X$ by iterating the previous steps.

3. Define the cosimplicial space $(\underline{R}X)^k := R^{k+1}X$.
4. The R**-completion of** $\underline{R}X$ is the *totalisation* $R_\infty X := \mathrm{Tot}\underline{R}X = \int_\Delta \underline{R}X^{\Delta[-]}$
5. Totalisation can be seen as the limit of a tower of fibrations, we call it the R-tower $(R_sX)_s$. We will often use that $R_\infty X = \lim R_sX$.

- 2.3.2. The use of the $R \otimes -$ symbol above is justified by the fact that R is a monad; indeed, for R a ring this will just be the usual tensor product.

The previous construction takes us through the following categories **sSet** → **csSet** → **Pro**(**sSet**) in a rather *ad hoc* way, as the intermediate constructions do not satisfy any universal property. In [Isa05], Isaksen avoids taking these intermediate steps by working directly in the category **Pro**(**sSet**), and proves that in certain model structure, the Bousfield-Kan R-completion is characterised as a fibrant replacement. We will now state the main results concerning us.

Construction 2.5. The **strict model structure** of **Pro**(**sSet**):

- **Cofibrations** are levelwise essentially cofibrations.
- **Weak equivalences** are levelwise essentially weak equivalences.

Proposition 2.6. *[Isa05, Theorem 2.2] Let K be a set of fibrant spaces. There exists a left proper simplicial model structure on the category of pro-spaces such that the cofibrations are the essentially levelwise cofibrations and such that a map $f : X \to Y$ is a weak equivalence if and only if*

$$\mathrm{Map}_{\mathrm{pro}}(Y, cA) = \mathrm{colim}_s \mathrm{Map}(Y_s, A) \to \mathrm{colim}_t \mathrm{Map}(X_t, A) = \mathrm{Map}_{\mathrm{pro}}(X, cA)$$

is a weak equivalence for every object A in K.

- 2.3.3. We can use the above proposition to find from the strict model structure two concrete model structures on **Pro**(**sSet**) that will encode (co)homological data. By setting the set of fibrant objects K above to be respectively the set of Eilenberg-MacLane spaces $K(R, n)$, and the set of Eilenberg-MacLane spaces $K(M, n)$ for a M an R-module whose generating set has cardinality at most λ a fixed infinite cardinal, we get:

Construction 2.6. [Isa05, theorem 6.3] The **cohomological model structure** L^R**Pro**(**sSet**) where **cofibrations** are levelwise essentially cofibrations and **weak equivalences** are levelwise $R-$**cohomology weak equivalences**: a morphism of profinite spaces $f : X \to Y$ such that the induced $f^* : H^*(Y, R) \to H^*(X, R)$ is an isomorphism of groups.

Construction 2.7. [Isa05, theorem 6.7] The **homological model structure** L_R**Pro**(**sSet**) where **cofibrations** are levelwise essentially cofibrations and **weak equivalences** are levelwise $R-$**homology weak equivalences**: a morphism of profinite spaces $f : X \to Y$ such that the induced $f_* : H_*(X, R) \to H_*(Y, R)$ is an isomorphism of groups.

Definition 2.8. [Isa05, definition 7.1] Let X be a pro-space. The **cohomological R-completion** X^{R-c} of X is a fibrant replacement for X in the cohomological model structure. The **homological R-completion** X^{R-h} of X is a fibrant replacement for X in the homological model structure

The conceptual upshot, Isaksen discusses, is that (co)fibrant replacements in these model structures will remember (co)homologies with R-coefficients and forget everything else.

The above definitions of R-completion are related to Bousfield-Kan's original construction by the following proposition:

Proposition 2.7. *[Isa05, Proposition 7.3] Let X be a pro-object in the category of pointed connected spaces, and let I be the indexing category of X. Construct a new pro-space Y with indexing category $I \times \mathbb{N}$ by defining $Y_{s,n}$ to be the n^{th} Postnikov section $P_n R_n X_s$ of the n^{th} stage of the Bousfield-Kan R-tower for X_s. Then the strict fibrant replacement Y^f of Y is a fibrant replacement for X in the R-homological model structure.*

Remark 2.3.4. The above does not give strictly speaking an equivalence between fibrant replacements in the homological model structure and the R-completion, but with Postnikov sections of the R-tower. As Isaksen discusses, we could solve this by putting stronger conditions on the fibrant replacement [Isa05, Remark 7.4]. However, proposition 2.7 will be sufficient for us: as it will become clear in the following, we will take the homotopy limit on the both the Postnikov section and R-tower indexes, which is equivalent to the R-completion.

Isaksen shows that for an arbitrary ring R, a map between two spaces $f : X \to Y$ that is a weak homology equivalence will also be a weak cohomology equivalence. That is, the cohomological model structure is a localisation of the homological model structure. Conversely, Horel shows that in the case of $R = \mathbb{Z}$ (more generally, in the case that the ring is a principal ideal domain):

Proposition 2.8. *[Hor22, Proposition 6.2] Let X be a connected space of finite type. Let $X \to \{Y_\alpha\}$ be a fibrant replacement of X in $L^{\mathbb{Z}}\mathbf{Pro}(\mathbf{sSet})$. Then the map $X \to \{Y_\alpha\}$ is a homology equivalence and so is a fibrant replacement in $L_{\mathbb{Z}}\mathbf{Pro}(\mathbf{sSet})$.*

This result is useful as we are working with cohomological quasi-isomorphims, and is used in the proof of theorem B below.

Theorem B. Let X be a connected space of finite type, then map

$$X \mapsto \langle \mathbb{Z}^X \rangle$$

is weakly equivalent to the Bousfield-Kan $\mathbb{Z}$-completion $\mathbb{Z}_\infty X$.

Proof. Let X be a connected space of finite type, and let $X \to \{Y_\alpha\}$ be a fibrant replacement in $L^{\mathbb{Z}}\mathbf{Pro}(\mathbf{sSet})$, then by proposition 2.8 it is also a fibrant replacement in $L_{\mathbb{Z}}\mathbf{Pro}(\mathbf{sSet})$. By results in [Isa05] we can replace the fibrant replacement $\{Y_\alpha\}$ by its connected components $\{Z_\alpha\}$ and it remains a fibrant replacement, furthermore, they turn out to be nilpotent too.

By proposition 2.7 we have a second fibrant replacement $X \to \{P_n R_n X\}$ that is weakly equivalent to the first. Thus, arriving to the following equivalence

Finally, as $X \to \{Z_\alpha\}$ is a $\mathbb{Z}$-cohomology equivalence we get the cohomology equivalence in cosimplicial binomial rings of $\mathrm{hocolim}_\alpha \mathbb{Z}^{Z_\alpha} \to \mathbb{Z}^X$. We apply the right adjoint $\langle - \rangle$ and get

$$\langle \mathbb{Z}^X \rangle \simeq \langle \mathrm{hocolim}_\alpha \mathbb{Z}^{Z_\alpha} \rangle \simeq \mathrm{holim}_\alpha \mathbb{Z}_\alpha Z \simeq \mathbb{Z}_\infty Z$$

Where the second equivalence comes from theorem A, as the spaces Z_α are nilpotent. The second weak equivalence can be deduced from Milnor's exact sequence:

$$\begin{tikzcd}
0 \arrow[r] & \lim^1(\pi_{i+1}(\mathbb{Z}_\alpha Z), z) \arrow[r] \arrow[d] & \pi_i(\lim \mathbb{Z}_\alpha Z), z) \arrow[r] \arrow[d] & \lim \pi_i(\mathbb{Z}_\alpha Z, z) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \lim^1(\pi_{i+1}(P_\alpha R_\alpha Z), z) \arrow[r] & \pi_i(\lim P_\alpha R_\alpha Z), z) \arrow[r] & \lim \pi_i(P_\alpha R_\alpha Z, z) \arrow[r] & 0
\end{tikzcd}$$

Here, we use the fact that $\mathbb{Z}_\infty \cong \lim \mathbb{Z}_\alpha Z$ discussed in construction 2.4. For a fixed i, we want the central vertical arrow to be an isomorphism; to check this we show that the other two vertical ones are. Indeed, by the definition of a Postnikov tower, for the first index $\alpha \geq i$, the homotopy groups $\pi_i(\mathbb{Z}_\alpha Z, z) \cong \pi_i(P_\alpha \mathbb{Z}_\alpha Z, z)$ are isomorphic for each $\mathbb{Z}_\alpha Z$, so their limit over α will too be isomorphic. Similarly for the first vertical arrow; so we get the desired result. ☐

3 NORI MOTIVES

3.1 A TRADITIONAL TANNAKIAN APPROACH

We briefly survey Nori's universal abelian construction and Nori motives. **For the whole of this section, let R be a Dedekind domain**. Recall that Nori's construction is purely algebro-categorical, in the sense that it arises as a solution to the following problem:

$$\begin{tikzcd}
& \mathcal{N}(Q,\rho) \arrow[d, dashed, "\Phi F"] \arrow[ddr, "ff_\rho"] & \\
& \mathcal{A} \arrow[dr, "\varphi"] & \\
Q \arrow[uur, "\tilde{\rho}"] \arrow[ur, "F"] \arrow[rr, "\rho"'] & & R-\mathbf{mod}
\end{tikzcd}$$

- 3.1.1. Given a quiver representation $\rho : Q \to R-\mathbf{mod}$, can we find a category $\mathcal{N}(Q,\rho)$ such that given any R-linear, faithful, exact abelian category representation $\varphi : \mathcal{A} \to R-\mathbf{mod}$ through which ρ factorises, there exists an R-linear functor $\Phi F : \mathcal{N}(Q,\rho) \to \mathcal{A}$ unique up to unique isomorphism making the above diagram commute up to isomorphism?

- 3.1.2. An elegant way to solve this problem would be to have a **functorial construction** $\mathcal{N}(-,-)$ and an **equivalence of categories** $\mathcal{A} \simeq \mathcal{N}(\mathcal{A},\varphi)$ for all R-linear faithful exact abelian category representations $\varphi : \mathcal{A} \to R-\mathbf{mod}$. As, then, we would get our desired unique up to unique isomorphism functor ΦF for free

$$\begin{tikzcd}
Q \arrow[r, "\widehat{\rho}"] \arrow[d, "F"'] \arrow[rr, bend left=50, "\rho" pos=0.8, shift={(0,0)}, to=2-3] & \mathcal{N}(Q,\rho) \arrow[dl, dashed, "\Phi F"'] \arrow[d, "\mathcal{N}F"] \arrow[dr, "ff_\rho"] & \\
& & R-\mathbf{mod} \\
\mathcal{A} \arrow[r, shift right, "\widehat{\varphi}"'] \arrow[urr, bend right=50, "\varphi"'] & \mathcal{N}(\mathcal{A},\varphi) \arrow[l, shift right, red] \arrow[ur, "ff_\varphi"'] &
\end{tikzcd}$$

- 3.1.3. This class of problems falls under the umbrella of Tannaka reconstruction. Usual Tannaka reconstruction establishes that given a field k, a rigid, abelian tensor category $\mathcal{A}$, such that $\underline{\mathrm{End}}(\mathbf{1}) = k$, and a faithful exact k-linear representation $\varphi : \mathcal{A} \to \mathbf{Vec}_k$, the following equivalence holds: $\mathcal{A} \simeq \mathbf{Rep}\,(\underline{\mathrm{Aut}}(\varphi))$ [Del+82, pp. 130, 131, theorem 2.11]. We are looking for a weaker version of this statement, whence the interest in endomorphisms (instead of automorphisms) in the following.

Construction 3.1. Let $\rho : Q \to R-\mathbf{mod}$ be a quiver representation, and let I be the category of (full) finite subquivers Q_i of Q, and morphisms are inclusions. We construct the **endomorphisms of a representation of a finite subquiver** Q_i as the *end* object

$$\underline{\mathrm{End}}(\rho|_{Q_i}) := \int_{Q_i} (\underline{\mathrm{Hom}}(\rho-, \rho-))$$

The **endomorphisms of the representation** ρ is the pro-algebra

$$\underline{\mathrm{End}}(\rho) := \lim_I \underline{\mathrm{End}}(\rho|_{Q_i})$$

Definition 3.2. [Nor, Paragrapgh 1.2.1] Let $\rho : Q \to R-\mathbf{mod}$ be a quiver representation, we define **Nori's universal abelian category** as the category of finitely generated R-modules under the action of the ρ-endomorphisms

$$\mathcal{N}(Q, \rho) := \underline{\mathrm{End}}(\rho)-\mathbf{mod} = \left(\lim_I \underline{\mathrm{End}}(\rho|_{Q_i})\right)-\mathbf{mod}$$

- 3.1.4. An object in this category is a triple $(M, Q_M, \alpha_M : \underline{\mathrm{End}}(\rho|_{Q_M}) \to \underline{\mathrm{End}}(M))$, where M is in $R-\mathbf{mod}$, Q_M is a finite subquiver of Q, α_M is an action of $\underline{\mathrm{End}}(\rho|_{Q_M})$ on M. Given two triples $(M, Q_M, \alpha_M), (M', Q'_M, \alpha_{M'})$, an $R-$module morphism $f : M \to M'$ is a morphism in this category if and only if there exists a finite subquiver P such that f is an $\underline{\mathrm{End}}(\rho|_P)-\mathbf{mod}$ morphism.

Theorem 3.1. *(Nori's reconstruction theorem)[Nor, Proposition 1.10] Let $\mathcal{A}$ be an R-linear abelian category, let $\varphi : \mathcal{A} \to R-\mathbf{mod}$ be a faithful exact R-linear representation. Then, the canonical lift $\widehat{\varphi} : \mathcal{A} \to \mathcal{N}(\mathcal{A}, \varphi)$ is an equivalence of categories.*

Theorem 3.2. *(Nori) [Nor, Theorem 1.6] The construction $\mathcal{N}(Q, \rho)$ solves the problem posed in paragraph 3.1.1.*

We can now restrict to the case we were originally interested in: where our quiver keeps track of varieties and its representation is a cohomology theory.

Construction 3.3. The quiver of relative varieties over k, $Q(k)$, consists of:

- Objects are 4-tuples (X, Y, n, i), where X is variety over k, and Y is a closed subvariety of X and $n, i \in \mathbb{Z}$.
- Edges are of three types for all $n \in \mathbb{Z}$
    1. For each morphism $f : X \to X'$, with $Y \hookrightarrow X, Y' \hookrightarrow X'$, and $f(Y) \subseteq Y'$, we have a *pullback* edge $f^* : (X', Y', n, i) \to (X, Y, n, i)$.
    2. For each triple $Z \hookrightarrow Y \hookrightarrow X$ of closed subvarieties of X for we have a *boundary* edge $\delta : (Y, Z, n, i) \to (X, Y, n + 1, i)$.
    3. for each $n, i \in \mathbb{Z}$, a *twist* edge $\xi : (X \times \mathbb{G}_m, X \times \{1\} \cup Y \times \mathbb{G}_m, n, i) \to (X \times \mathbb{G}_m, X \times \{1\} \cup Y \times \mathbb{G}_m, n - 1, i - 1)$.

Construction 3.4. We define the quiver representation $\mathcal{H}$ to be

$$\mathcal{H} : Q(k) \to R\text{−}\mathbf{mod}$$

$$(X, Y, n, i) \mapsto H^n(X, Y, R)(i)$$

where $H^n(X, Y, R)$ is the singular relative cohomology. The twist (i) means tensoring $|i|$ times by the **Lefschetz motive** $H^1(\mathbb{G}_{m,k}(\mathbb{C}), 1, R)$ when $i < 0$ and by its linear dual, the **Tate motive**, when $i \geq 0$. Quiver morphisms are sent to their obvious realisations.

Definition 3.5. The **category of Nori motives** $\mathcal{M}_{\text{Nori}}$ over k with coefficientes in R is the universal abelian category $\mathcal{N}(Q(k), \mathcal{H})$.

$$\begin{tikzcd}
 & \mathcal{M}_{\text{Nori}} \arrow[dr, "ff"] & \\
Q(k) \arrow[ur, "\hat{\mathcal{H}}"] \arrow[rr, "\mathcal{H}"'] & & R\text{−}\mathbf{mod}
\end{tikzcd}$$

- 3.1.5. Each $E_i := \underline{\text{End}}(\mathcal{H}|_{Q_i})$ is finitely generated and projective as an R-module [Jos16, Corollary 3.8], so their dual $A_i := \underline{\text{Hom}}(E_i, R)$ is a finitely generated projective R-coalgebra, and $E_i\text{−}\mathbf{mod} \simeq A_i\text{−}\mathbf{comod}$ [HM17, Lemma 7.5.4]. As the E_i are projective, $\underline{\text{Hom}}(-, R)$ commutes with colimits, so we get that $\underline{\text{Hom}}(\underline{\text{End}}(\mathcal{H}), R) := \underline{\text{Hom}}(\lim_I E_i, R) \cong \text{colim}_I A_i$. Given that $E_i\text{−}\mathbf{mod} \simeq E_i\text{−}\mathbf{mod}^{\text{op}}$, we can consider the colimit of sheaves $\text{colim}_I \text{Hom}(E_i, -) \cong \text{colim}_I \text{Hom}(-, E_i)$, this is a colimit of representables in the category of presheaves, so it exists. When we evaluate it at R, $\text{colim}_I A_i$ comes with a flat Hopf algebra structure. We denote it by H^N. We conclude that $\mathcal{M}_{\text{Nori}} := \underline{\text{End}}(\mathcal{H})\text{−}\mathbf{mod} \simeq \mathbf{Comod}_{H^N} = \mathcal{M}^{\text{op}}_{\text{Nori}}$.

Definition 3.6. We call H^N as defined above **Nori's Hopf algebra**, and define the **motivic Galois group** $G_k(R)$ of the field k to be the group scheme $\text{Spec}(H^N)$.

- 3.1.6. The motivation for this definition is the following that by general theory of algebraic groups, the category of representations of one is equivalent to comodules over a Hopf algebra. In our case, we get the following equivalences:

$$\mathcal{M}_{\text{Nori}} := \underline{\text{End}}(\mathcal{H})\text{−}\mathbf{mod} \simeq \mathbf{Comod}_{H^N} \simeq \mathbf{Rep}(G_k(R))$$

At the end of the day, Nori motives are but a universal (co)homology theory, so it would be nice to have some filtering result that made it easier to compute these motives similarly to how we do with ordinary (co)homology. Indeed, we end the section by stating one such result:

Lemma 3.3. *[Nor, Theorem 2.1](Nori's Basic Lemma) Let X be an affine variety of dimension n, and $W \subset X$ a closed subvariety of dimension $\leq n - 1$. Then there* *exists a closed subvariety $W \subset Z \subset X$ of dimension $\leq n-1$ such that $H_i(X, Z, R)$ is a free abelian group concentrated in degree n*

Definition 3.7. Let X be an affine variety over k, then a **filtration** of X is an increasing sequence $\{F_iX\}_{i\in\mathbb{Z}}$ of closed subvarieties such that

1. The dimension of each element is $\dim(F_iX) \leq i$.
2. There exists some n such that $F_nX = X$.

A filtration will be said to be **very good** if for every i, all pairs (FX_i, FX_{i-1}) are either such that:

- The subvariety F_iX is affine of dimension i

  or

- the subvariety F_{i-1} is of dimension $\leq i-1$, $F_iX\backslash F_{i-1}$ is smooth, and $H_*(F_iX, F_{i-1}; R)$ is a free abelian group concentrated in degree i

  or

- the subvarieties $F_iX = F_{i-1}$ are equal, affine and of dimension $< i$.

Corollary 3.4. *Let X be an affine variety, and let $\{F_iX\}$ be a filtration of X, then there exists a very good filtration $\{G_j\} \subseteq \{F_iX\}$.*

Proof. We go by induction and apply Nori's Basic Lemma (see [CG17, Collorary 6.2]). □

3.2 A MODERN CONJECTURAL APPROACH

Remark 3.2.1. A reader who is comfortable with abruptly jumping from $\mathbf{Comod}_{H^N}$ the category of *finitely generated* comodules over H^N to the category $\mathbf{CoMod}_{H^N}$ of comodules over H^N might want to skip this subsection.

Although the results in the previous subsection are well established, they are limited in two ways: in the first place, we have to work with finitely generated objects, secondly, the mathematical presentation might strike one as streamlined enough. As to try to amend these two objections, I will very superficially present an alternative characterisation of Nori motives using the modern work of Ayoub, Choudhury, and Gallauer [Ayo14][CG17]. The goal here is to justify the use of the full category $\mathbf{coMod}_{H^N}$ of comodules over H^N instead of only finitely generated ones. I do not aspire to be comprehensive.

Definition 3.8. We fix an embedding $k \hookrightarrow \mathbb{C}$, and a coefficient ring R, we denote by $\mathbf{DA}(k, R)$ the conjectural **triangulated category of motives**.

This category has a Betti realisation functor $\mathrm{Bti}^* : \mathbf{DA}(k,r) \to \mathbf{D}(R)$ to the derived category of R-modules, taking a motive to its cohomology. It has a right adjoint Bti_*. The endofunctor $\mathrm{Bti}^*\mathrm{Bti}_*(R)$ applied to the coefficient ring is a Hopf algebra in the derived category $\mathbf{D}(R)$. We call it **Ayoub's Hopf algebra** $\mathbb{H}^A$. It is known that its homology groups are zero in negative degree, and it is conjectured by Ayoub that:

Conjecture 3.1. Ayoub's Hopf algebra $\mathbb{H}^A$ is concentrated in degree 0.

- 3.2.2. If this was so, then the derived category of comodules over $\mathbb{H}^A$ would be equivalent to the category $\mathbf{CoMod}_{H_0(\mathbb{H}^A,R)}$

Theorem 3.5. *[CG17, Theorem 9.1] Nori's Hopf algebra H^N is isomorphic to the 0^{th} homology of Ayoub's Hopf algebra $H_0(\mathbb{H}^A, R)$. In particular, we have an equivalence of categories* $\mathbf{CoMod}_{H_0(\mathbb{H}^A,R)} \simeq \mathbf{CoMod}_{H^N}$.

In light of this theorem and the conjectural discussion of paragraph 3.2.2, it would then be sensible to redefine Nori motives as:

Definition 3.9. (Conjectural definition of Nori motives) $\mathcal{M}_{\text{Nori}} := \mathbf{CoMod}_{H^N}$.

Remark 3.2.3. The above discussion is not really dependent on the essential image of the cohomology functor, which for algebraic varieties will take values in finitely generated R-modules always, but on the approach one takes in building a theory of motives. For a comparison between the two approaches discussed in this section see [Ayo14, "Comparaison avec le théorème Tannakien de Nori"]

4 CONJECTURES AND MODELS

4.1 AN UNEXPECTED AFFAIR: CONJECTURES

In the following we will make use of the conjectural definition we just stated $\mathcal{M}_{\text{Nori}} := \mathbf{CoMod}_{H^N} \simeq \mathbf{Rep}(G_k(R))$. Let us do some technical housekeeping before we state Horel's conjectures.

Lemma 4.1. *The Hopf algebra H^N has a binomial ring structure.*

Proof. Let $\mathbb{F}_p$ be the finite field of p elements, then the following isomorphism holds: $\mathbb{F}_p \otimes H^N \cong \mathbb{F}_p^{\mathrm{Gal}(\bar{k}/k)}$ [Jos16, pp.48, 49]. Hence, for all x in H^N, $x^p - x$ will be zero, so H^N has a binomial ring structure. ☐

- 4.1.1. We conclude that the in the category of representations of the motivic Galois group, the coaction $M \to M \otimes H^N$ will be preserved even when restricted to those representations with a binomial ring structure. This would not necessarily be the case if H^N was not already binomial. So not only can we consider the full subcategory of representations with a ring structure $\mathbf{Ring}(\mathbf{CoMod}_{H^N})$, abbreviated as $\mathbf{Ring}_{H^N}$, but also that of representations which have a *binomial* ring structure $\mathbf{BRing}(\mathbf{CoMod}_{H^N})$, abbreviated as as $\mathbf{BRing}_{H^N}$. Moreover, we can upgrade the adjunctions between abelian groups, rings and binomial rings to these categories and then extend them levelwise to the cosimplicial setting, obtaining the following diagram:

$$\begin{tikzcd}
\mathbf{cBRing} \arrow[r, harpoon, shift left] \arrow[d, harpoon, shift left] & \mathbf{cRing} \arrow[l, harpoon, shift left, "\mathrm{Bin}^U"] \arrow[r, harpoon, shift left] \arrow[d, harpoon, shift left] & \mathbf{cAb} \arrow[l, harpoon, shift left, "\mathrm{Sym}"] \arrow[d, harpoon, shift left, "\otimes H^N"] \\
\mathbf{cBRing}_{H^N} \arrow[r, harpoon, shift left] \arrow[u, harpoon, shift left] & \mathbf{cRing}_{H^N} \arrow[l, harpoon, shift left] \arrow[r, harpoon, shift left] \arrow[u, harpoon, shift left] & \mathbf{cCoMod}_{H^N} \arrow[l, harpoon, shift left] \arrow[u, harpoon, shift left]
\end{tikzcd}$$

Notice that in the above diagram the vertical adjunctions induced by $\otimes H^N$ and the forgetful functor have the latter as the *left* adjoint and the former as the *right* adjoint (due to the comodule structure).

Definition 4.1. We define the functor

$$\mathrm{Spec} : \mathbf{cBRing}^{\mathrm{op}} \to \mathbf{Fun}(\mathbf{BRing}, \mathbf{sSet})$$

$$A \mapsto \mathbb{R}\mathrm{map}_{\mathbf{cBRing}}(A, -)$$

Where the argument of $\mathrm{Spec}(A)(R)$ is a constant cosimplicial binomial ring R.

- 4.1.2. For a simplicial set X, we denote by X^{bin} the functor $\mathrm{Spec}(\mathbb{Z}^X)(-)$. Following Töen [**Toen20**], Horel calls $X^{\mathrm{bin}}(R)$ an **affine binomial stack**.

For the constant cosimplicial Nori Hopf algebra H^N, the functor $\mathrm{Spec}(H^N)$ gives us, unsurprisingly, the motivic Galois group $G_k(R)$ when we evaluate it at R.

- 4.1.3. Finally, before stating our conjecture, I will recall the **analytification** of a finite type scheme X over $k \hookrightarrow \mathbb{C}$, X^{an}, in the sense of Serre [Ser56]. Let A be a finitely generated k-algebra, for the affine case, we consider the set of closed points of $\mathrm{Spec}(A)$ under the Zariski topology $\mathrm{MaxSpec}(A)$. This set is in bijection with the set of k-algebra morphisms $\{A \to k \hookrightarrow \mathbb{C}\}$. Choose generators $(a_1, ..., a_n)$ for A; there is a surjection $\mathbb{C}[x_1, ..., x_n] \to A$. This surjection induces an injection $\theta : \mathrm{MaxSpec}(A) \to \mathrm{MaxSpec}(\mathbb{C}[x_1, ..., x_n])$. The set $\mathrm{MaxSpec}(\mathbb{C}[x_1, ..., x_n])$ is in bijection with that of $\mathbb{C}$- algebra morphisms $\{\mathbb{C}[x_1, ..., x_n] \to \mathbb{C}\}$, which is isomorphic to $\mathbb{C}^n$. Hence, we can give it its usual metric topology, and so $\mathrm{MaxSpec}(A)$ also gets a subspace topology of $\mathbb{C}^n$ via the θ injection. We call $\mathrm{MaxSpec}(A)$ with this complex topology its analytification $\mathrm{Spec}(A)^{\mathrm{an}}$. This construction works well with gluing, so we can extend this to a finite type scheme X over k [Nee07, Ch.4].

We are now ready to establish the conjectural relationship that motivates the work in this section and fuses the world of integral homotopy theory à la Horel and Nori motives.

Conjecture 4.1. (Horel) Let k be a field with a fixed embedding $k \hookrightarrow \mathbb{C}$, and let $\mathbf{Var}_k$ be the category of varieties over k. Then there exists a functor $A : \mathbf{Var}_k \to \mathbf{cBRing}^{\mathrm{op}}_{H^N}$ such that the following diagram commutes up to weak equivalence.

$$\begin{tikzcd}
\mathbf{Var}_k \arrow[r, dashed, "A"] \arrow[d, "B"'] & \mathbf{cBRing}^{\mathrm{op}}_{H^N} \arrow[d, "W"] \\
\mathbf{sSet} \arrow[r, "\mathbb{Z}^{(-)}"'] & \mathbf{cBRing}^{\mathrm{op}}
\end{tikzcd}$$

where:

- For a variety X, the functor B gives the simplicial set corresponding to its analytification X^{an} seen as a topological space.
- The functor W forgets the H^N comodule structure.
- The conjectural functor A must agree with the functor C constructed by Choudhury-Gallauer [CG17, Section 6].

$$\begin{tikzcd}
\mathbf{Var}_k \arrow[r, "A"] \arrow[rrr, bend right, "C"'] & \mathbf{cBRing}^{\mathrm{op}}_{H^N} \arrow[r, "i"] & \mathbf{cCoMod}^{\mathrm{op}}_{H^N} \arrow[r, "N"] & \mathbf{Ch}^*(\mathbf{CoMod}^{\mathrm{op}}_{H^N})
\end{tikzcd}$$

  The functor C assigns to a variety X, the ind object $\mathrm{colim}_{F_*} H^*(X, F_*, \mathbb{Z})$, where

  - The colimit runs over the class of filtrations of X
  - $H^*(X, F_*, \mathbb{Z})$ is the cochain

$$H^0(F_0X^{\mathrm{an}}, , \mathbb{Z}) \to \ldots \to H^{n-1}(F_{n-1}X^{\mathrm{an}}, F_{n-2}X^{\mathrm{an}}, \mathbb{Z}) \to H^n(X^{\mathrm{an}}, F_{n-1}X^{\mathrm{an}}, \mathbb{Z})$$

- 4.1.4. As $A(X)$ is a comodule over H^N, we get a map $A(X) \to A(X) \otimes H^N$, so applying $\mathbb{R}\mathrm{map}_{\mathbf{cBRing}}(-,-)$ we obtain the following map

$$\mathbb{R}\mathrm{map}_{\mathbf{cBRing}}(A(X) \otimes H^N, -) \cong$$

$$\mathbb{R}\mathrm{map}_{\mathbf{cBRing}}(A(X), -) \times \mathbb{R}\mathrm{map}_{\mathbf{cBRing}}(H^N, -) \to \mathbb{R}\mathrm{map}_{\mathbf{cBRing}}(A(X), -)$$

Hence, for a variety X, $\mathrm{Spec}(H^N)(-) := G_k(-)$ acts on $\mathrm{Spec}(A(X))$. By the weak commutativity of the above diagram, we have $A(X) \xrightarrow{\sim} \mathbb{Z}^{B(X)}$; let us abuse notation and call $B(X)$ just X. Then, by the above discussion, we get that $X^{\mathrm{bin}}(R)$ has an action of $G_k(R)$, so it induces the following:

Conjecture 4.2. (Horel) Let X be a variety over $k \hookrightarrow \mathbb{C}$, and let x be a k-point, then the homotopy groups of its pointed affine binomial stack $\pi_*(X^{\mathrm{bin}}(R), x)$ have an action of the motivic Galois group $G_k(R)$, ie. they are Nori motives.

Remark 4.1.5. Furthermore, all the previous discussion can be extended to the general case where $\mathrm{Spec}(A)(-)$ takes any ring as an argument. As we mentioned in paragraph 2.1.1, there exists an adjunction $U : \mathbf{cBRing} \rightleftharpoons \mathbf{cRing} : \mathrm{Bin}_U$, where Bin_U is the right adjoint. Then for any ring R we can consider $\mathbb{R}\mathrm{map}_{\mathbf{cBRing}}(B, \mathrm{Bin}_U(R)) \cong \mathrm{map}_{\mathbf{cRing}}(U(B^c), R)$, where B^c is a cofibrant replacement in $\mathbf{cBRing}$.

4.2 Model Structures on Cosimplicial Comodules and Friends

- 4.2.1. In order do to homotopy theory, we would like to give these (sub)categories of representations of the motivic Galois group a model structure. However, Nori's motivic Galois group is relevant insofar as it is a binomial Hopf algebra, so we will give more general results: **In the following let H be a Hopf algebra which is flat as an R-module and binomial as a commutative ring**. We force H to be flat as to make the category $\mathbf{comod}_H$ not only abelian, but also Grothendieck (see [Wis75, Corollary 9]), and binomial to preserve the comodule structure up to $\mathbf{cBring}_H$ as discussed in paragraph 4.1.1. We use the work of Hovey in [Hov07] to find these model structures. Before going into it, let us define

Definition 4.2. Let $\mathcal{A}$ be an abelian category, a **cotorsion pair** $(\mathcal{C}, \mathcal{F})$ is a pair of subcategories satisfying the following conditions:

- An object A is in $\mathcal{C}$ if and only if for all objects B of $\mathcal{F}$ we have that $\mathrm{Ext}^1(A, B) = 0$.
- An object B is in $\mathcal{F}$ if and only if for all objects A of $\mathcal{C}$ we have that $\mathrm{Ext}^1(A, B) = 0$.

Hovey explores the general question of how to get a cotorsion pair from an abelian model category $\mathcal{M}$, and, more importantly for us, how to get a model structure on an abelian category from a cotorsion pair. It turns out that one may go both ways. However, in order to be able to recover a model structure that is both abelian and respects monoidal strcutures we require our category $\mathcal{A}$ to be Grothendieck like in our case. The fact that it is cofibrantly generated requires one more condition; luckily $\mathbf{cBRing}_{H^N}$ in particular will satisfy it.

Definition 4.3. An **abelian model category** is an complete and cocomplete abelian category $\mathcal{A}$ equipped with a model structure such that

1. A map is a cofibration if and only if it is a monomorphism with cofibrant cokernel.

2. A map is a fibration if and only if it is an epimorphism with fibrant kernel.

Theorem 4.2. *[Hov07] Let $\mathcal{A}$ be an abelian category, and let $(\mathcal{C}, \mathcal{F})$ be a cotorsion pair, then there exists an abelian model structure on $\mathcal{A}$ where:*

- ***Fibrations** are **epimorphisms** with **kernel** in $\mathcal{F}$*
- ***Cofibrations** are **monomorphisms** with **cokernel** in $\mathcal{C}$*

Example 4.4. Let $\mathcal{A}$ be any abelian category, we can take the "projective" cotorsion pair $(\mathcal{C} = \mathbf{Projectives}, \mathcal{F} = \mathcal{A})$. Similarly, we can construct the "injective" cotorsion pair $(\mathcal{C} = \mathcal{A}, \mathcal{F} = \mathbf{Injectives})$.

- 4.2.2. Consider the category of non-negative cochains of R-modules $\mathbf{Ch}^*(R)$, it has a well known projective Quillen model structure whose fibrations and cofibrations coincide with those of the projective cotorsion pair $(\mathcal{C} = \mathbf{Projectives}, \mathcal{F} = \mathbf{Ch}^*(R))$. As a model structure is completely determined by this information, we deduce that the weak equivalences of our cotorsion projective structure must be quasi-isomorphisms. One might wonder what these weak equivalences look like from the point of view of cotorsion pairs; this motivates the following definition:

Definition 4.5. Let $\mathcal{A}$ be an abelian category, and let $\mathcal{W}$ be a subcategory, we say $\mathcal{W}$ is **thick** if it is closed under retracts and whenever 2 out of 3 objects of a short exact sequence are in $\mathcal{W}$, so is the third.

Definition 4.6. Let $\mathcal{A}$ be an abelian category, a cotorsion pair $(\mathcal{E}, \mathcal{D})$ is **complete** if for all object X:

1. There exists E in $\mathcal{E}$, and D in $\mathcal{D}$ such that the following sequence is exact
$$0 \to E \to D \to X \to 0$$
2. There exists E in $\mathcal{E}$, and D in $\mathcal{D}$ such that the following sequence is exact
$$0 \to X \to E \to D \to 0$$

Theorem 4.3. *[Hov07, Theorem 2.5] Let $\mathcal{A}$ be a bicomplete abelian category, let $\mathcal{C}, \mathcal{F}$ be subcategories, and $\mathcal{W}$ a thick category. If the cotorsion pairs $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$, and $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ are complete, then there exists a unique abelian model structure compatible with that of theorem 4.2.*

- 4.2.3. Indeed, we now get a characterisation of acyclic maps from this relationship. A fibration (resp. cofibration) will be weak if its kernel (resp. cokernel) is in $\mathcal{F} \cap \mathcal{W}$ (resp. $\mathcal{C} \cap \mathcal{W}$). Weak equivalences will simply be all maps of the form $w = c_w \circ f_w$ where c_w, f_w are a weak cofibration and a weak fibration respectively.

Moreover, we can also go in the converse direction:

Lemma 4.4. *[Hov07, Lemma 2.4] Let $\mathcal{M}$ is be abelian model category and $\mathcal{W}$ is the subcategory of objects weakly equivalent to* 0*, then $\mathcal{W}$ is thick.*

- 4.2.4. We now go back to our test case of $\mathbf{Ch}^*(R)$. From lemma 4.4 we deduce that the thick category that matches the projective cotorsion pair is $\mathcal{W} =$ **Exact Complexes**. This will be useful in the following.

Let us now recall what $\mathbf{CoMod}_H$ the category of comodules over a binomial flat Hopf algebra H looks like:

- As a Hopf algebra, H is a coalgebra, particularly, it has a counit map $\varepsilon : H \to R$ and a codiagonal map $\Delta : H \to H \otimes H$.
- Objects consist of R-modules M with an H coaction $\rho : M \mapsto M \otimes H$ such that:
  1. It is coassociative: $(\rho \otimes \mathrm{id}_H) \circ \rho = (\mathrm{id}_M \otimes \Delta) \circ \rho : M \mapsto M \otimes H \otimes H$.
  2. It is counital: $(\mathrm{id}_M \otimes \varepsilon) \circ \rho = \mathrm{id}_M : M \mapsto M$
- Morphisms are just morphisms of R-modules respecting the above structure

Proposition 4.5. *There is a cofibrantly generated abelian model structure for the category of connective comodules over H, $\mathbf{Ch}^*(\mathbf{CoMod}_H)$, where **fibrations** are **levelwise epimorphisms**, and **weak equivalences** are **quasi-isomorphisms**.*

Proof. Consider the triple:
$(\mathcal{C}, \mathcal{F}, \mathcal{W}) = (\mathbf{Projectives}, \mathbf{Ch}^*(\mathbf{CoMod}_H), \mathbf{Exact\ Complexes})$.
Given the above description of the category of comodules over a Hopf algebra, our proposed pair $(\mathbf{Projectives}, \mathbf{Ch}^*(\mathbf{CoMod}_H))$ is a cotorsion pair, as this is checked in R-modules. It rests to verify that this cotorsion pair gives a model structure compatible with exact sequences as objects weakly equivalent to O. As our Hopf algebra H is flat, the functor $- \otimes H$ is exact (tensoring is always right exact). Hence, the exactness of sequences of theorem 4.3 will be preserved, and so our triple is a complete cotorsion pair giving us the desired model.

Finally, by the main result of [Sal22, Theorem 3.8], $\mathbf{Ch}^*(\mathbf{CoMod}_H)$ is Grothendieck and has enough projectives, and so by [Hov07, Proposition 3.2], the cotorsion pair model structure is cofibrantly generated. ☐

It could strike one as surprising to recover homological information in the form of weak equivalences from a cotorsion pair. Upon closer inspection, one realises this homological data is encoded in the definition of a the pair itself via the null Ext functor requirement.

Proposition 4.6. *There is a cofibrantly generated abelian model structure in* $\mathbf{cCoMod}_H$

where ***fibrations*** *are* ***levelwise epimorphisms****, and* ***weak equivalences*** *are* ***quasi-isomorphisms****.*

Proof. We transfer the model structure of $\mathbf{Ch}^*(\mathbf{CoMod}_H)$ by the Dold-Puppe correspondece $\mathbf{Ch}^*(\mathbf{CoMod}_H) \simeq \mathbf{cCoMod}_H$. $\square$

Following Horel's strategy, we will transfer this structure along the forgetful adjoints $\mathbf{cBRing}_H \to \mathbf{cRing}_H \to \mathbf{cCoMod}_H$

Proposition 4.7. *There exists a transferred cofibrantly generated model structure on* $\mathbf{cRing}_H$ *such that* ***fibrations*** *are* ***epimorphisms*** *and* ***weak equivalences*** *are* ***quasi-isomorphisms****.*

Proof. We will transfer the model structure of $\mathbf{cCoMod}_H$ along the forgetful functor. Indeed, the model structure is cofibrantly generated and every object is fibrant, so can attempt to apply the path object argument. We consider for a cosimplicial comodule X the object X^I given by

$$[n] \mapsto (X^n)^{\Delta([n],[1])} = \Pi_{n+2} X^n$$

We must check that the following holds:

$$X \xrightarrow{\sim} X^I \twoheadrightarrow X \times X$$

The second arrow is clearly an epimorphism, which is just a surjection of abelian groups. The first map is also a quasi-isomorphism, as it is in abelian groups. The only remaining thing to check is that this path object construction respects the comodule structure. Indeed, as it is defined levelwise by Cartesian products, we recover the comodule structure from that of each of the product's components. $\square$

Proposition 4.8. *There exists a transferred cofibrantly generated model structure on* $\mathbf{cBRing}_H$ *where* ***fibrations*** *are* ***epimorphisms*** *and* ***weak equivalences*** *are* ***quasi-isomorphisms****.*

Proof. The proof goes just as the previous one. Note that here the fact that H is binomial is relevant, as it assures that the coaction is preserved by the functor Bin^U. $\square$

Corollary 4.9. *There exists a cofibrantly generated model structures on the category of cosimplicial representations of the motivic Galois group* $\mathbf{cRep}(G_k(R))$*,* $\mathbf{cRing}_{H^N}$*, and* $\mathbf{cBRing}_{H^N}$ *where* ***fibrations*** *are* ***epimorphisms*** *and* ***weak equivalences*** *are* ***quasi-isomorphisms****.*

Proof. We apply propositions 4.6, 4.7, 4.8 with $H = H^N$. $\square$

References

[Ayo14] Joseph Ayoub. “L’algèbre de Hopf et le groupe de Galois motiviques d’un corps de caractéristique nulle, I”. In: *Journal für die reine und angewandte Mathematik (Crelles Journal)* 2014.693 (2014), pp. 1–149. DOI: `doi:10.1515/crelle-2012-0089`.

[BK72] A.K. Bousfield and D.M. Kan. *Homotopy Limits, Completions and Localizations*. Lecture Notes in Mathematics. Springer, 1972. ISBN: 9783540061052. URL: `https://books.google.es/books?id=6Mw2plpWo2sC`.

[CG17] Utsav Choudhury and Martin Gallauer Alves de Souza. “An isomorphism of motivic Galois groups”. In: *Advances in Mathematics* 313 (2017), pp. 470–536. ISSN: 0001-8708. DOI: `https://doi.org/10.1016/j.aim.2017.04.006`. URL: `https://www.sciencedirect.com/science/article/pii/S000187081630216X`.

[Del+82] Pierre Deligne et al. *Hodge Cycles, Motives and Shimura varieties*. Lecture Notes in Math. Springer-Verlag, 1982.

[Hor22] Geoffroy Horel. “Binomial Rings and Homotopy Theory”. In: *Preprint* 693 (2022), pp. 1–21. DOI: `https://arxiv.org/pdf/2211.02349.pdf`.

[Hov07] Mark Hovey. *Cotorsion pairs and model categories*. 2007. arXiv: `math/0701161 [math.AT]`.

[HM17] Annette Huber and Stefan Müller-Stach. *Periods and Nori Motives*. A Series of Modern Surveys in Mathematics. Springer, 2017. ISBN: 978-3-319-50925-9.

[Isa05] Daniel C. Isaksen. “Completions of pro-spaces”. In: *Math.Z.* 250 (2005), pp. 113–143. DOI: `https://doi.org/10.1007/s00209-004-0745-x`.

[Jos16] Peter Jossen. *On the relation between Galois groups and motivic Galois groups*. Last accessed July 11, 2023. 2016. URL: `http://www.jossenpeter.ch/PdfDvi/MoFGandGalois.pdf`.

[Man06] Michael A. Mandell. “Cochains and homotopy type”. In: *Publications mathématiques de l’IHÉS* 103.1 (June 2006), pp. 213–246. DOI: `10.1007/s10240-006-0037-6`. URL: `https://doi.org/10.1007%5C%2Fs10240-006-0037-6`.

[Nee07] Amnon Neeman. *Algebraic and Analytic Geometry*. London Mathematical Society Lecture Note Series. Cambridge University Press, 2007. DOI: `10.1017/CBO9780511800443`.

[Nor] Madhav Nori. *TIFR notes on motives (unpublished)*. Last accessed July 11, 2023. URL: `https://web.archive.org/web/20160922233016/http://www.arithgeo.ethz.ch/alpbach2011/Nori_TIFR`.

[Sal22] Andrew Salch. “Graded comodule categories with enough projectives”. In: *Glasgow Mathematical Journal* (2022), pp. 1–15. DOI: 10.1017/S0017089522000234.

[SS00] Stefan Schwede and Brooke E. Shipley. “Algebras and Modules in Monoidal Model Categories”. In: *Proceedings of the London Mathematical Society* 80.2 (Mar. 2000), pp. 491–511. DOI: 10.1112/s002461150001220x. URL: https://doi.org/10.1112%2Fs002461150001220x.

[Ser56] Jean-Pierre Serre. “Géométrie algébrique et géométrie analytique”. In: *Ann. Inst. Fourier (Grenoble)* 6 (1955/56), pp. 1–42. ISSN: 0373-0956,1777-5310. URL: http://aif.cedram.org/item?id=AIF_1955__6__1_0.

[Toë20] Bertrand Toën. “Le problème de la schématisation de Grothendieck revisité”. In: *Épijournal de Géométrie Algébrique* 4 (2020). URL: https://hal.science/hal-02361473v2.

[Wis75] Manfred B. Wischnewsky. “On linear representations of affine groups. I.” In: *Pacific Journal of Mathematics* 61.2 (1975), pp. 551–572.

[Xan17] Qimh Richey Xantcha. *Binomial Rings: Axiomatisation, Transfer and Classification*. 2017. arXiv: 1104.1931 [math.RA].